

Hawking radiation for Dirac spinors on the \mathbb{RP}^3 geon

Paul Langlois*

*School of Mathematical Sciences, University of Nottingham
Nottingham NG7 2RD, UK*

November 2004

Abstract

We analyse the Hawking(-Unruh) effect for a massive Dirac spinor on the \mathbb{Z}_2 quotient of Kruskal spacetime known as the \mathbb{RP}^3 geon. There are two distinct Hartle-Hawking-like vacua, depending on the choice of the spin structure, and suitable measurements in the static region (which on its own has only one spin structure) distinguish these two vacua. However, both vacua appear thermal in the usual Hawking temperature to certain types of restricted operators, including operators with support in the asymptotic future (or past). Similar results hold in a family of topologically analogous flat spacetimes, where we show the two vacua to be distinguished also by the shear stresses in the zero-mass limit. As a by-product, we display the explicit Bogolubov transformation between the Rindler-basis and the Minkowski-basis for massive Dirac fermions in four-dimensional Minkowski spacetime.

1 Introduction

It has been known for nearly thirty years that a black hole formed by star collapse will radiate thermally at the Hawking temperature [1]. This discovery led to black hole thermodynamics and in particular to the acceptance of one quarter of the area [2] as the physical entropy of the hole. It was realised shortly after the collapsing star analysis that the same temperature and entropy can also be obtained by considering the Kruskal-Szekeres extension of Schwarzschild and on it a quantum state that describes a black hole in thermal equilibrium with its environment [3, 4, 5, 6, 7, 8, 9]. The defining characteristics of this Hartle-Hawking state are that it is regular everywhere on the Kruskal manifold and invariant under the continuous isometries [8, 9].

From the physical point of view, a puzzling feature of the Hartle-Hawking state is its reliance on the whole Kruskal manifold. The manifold has two static regions, causally disconnected from each other and separated by a bifurcate Killing horizon, but the thermal properties manifest themselves when the state is probed in only one of the static regions. To explore the significance of the second exterior region, Louko and Marolf [10] investigated scalar field quantisation on the spacetime known in the terminology of [11] as the \mathbb{RP}^3 geon (for earlier work on the classical properties

*Electronic Address: pmxppl@nottingham.ac.uk

of this spacetime, see [12, 13]). The \mathbb{RP}^3 geon is a \mathbb{Z}_2 quotient of Kruskal, it is space and time orientable, it contains a black and white hole, but it only has one static region, isometric to standard exterior Schwarzschild. It was shown in [10] that the Hartle-Hawking like quantum state on the geon does not appear thermal to all observers in the exterior region, but it does appear thermal in the standard Hawking temperature when probed by suitably restricted operators. In particular, the state appears thermal in the standard temperature to every operator far from the hole and with support at asymptotically late (or early) Schwarzschild times.

The purpose of this paper is to extend the scalar field analysis of [10] to massive Dirac fermions. The main new issue with fermions is that while exterior Schwarzschild and Kruskal both have spatial topology $\mathbb{R} \times S^2$ and hence a unique spin structure, the \mathbb{RP}^3 geon has spatial topology $\mathbb{RP}^3 \setminus \{\text{point}\}$ and admits two inequivalent spin structures. The geon thus has *two* Hartle-Hawking like states for fermions, one for each spin structure. Our first aim is to examine whether these states appear thermal when probed in the exterior region: We shall find that they do, in a limited sense similar to what was found for the scalar field in [10]. Our second aim is to examine whether these two states can be distinguished by observations limited to the exterior region. We shall find that they can be in principle distinguished by suitable interference experiments: The states contain Boulware-type excitations in correlated pairs, and the spin structure affects the relative phase between the members of each pair. This means that the restriction of the Hartle-Hawking type state to the geon exterior not only tells that the classical geometry behind the horizons differs from Kruskal but also is sensitive to a quantisation ambiguity whose existence cannot be deduced from the exterior geometry. In this sense, probing the quantum state in the exterior region reveals in principle both classical and quantum information from behind the horizons. How this information might be uncovered in practice, for example by particle detectors with a local coupling to the fermion field, presents an interesting question for future work.

We analyse the same issues also on a family of Rindler spaces whose topology mimics that of the geon [10]. While the results are qualitatively similar to those on the geon, the effects of the spin structure appear in a much more transparent form, and these Rindler spaces thus offer a testing ground for localised particle detector models that aim to resolve the phase factors determined by the spin structure. We further compute the Rindler-space stress-energy tensor explicitly in the massless limit, showing that on the geon-type Rindler space the spatial orientation determined by the spin structure can be detected from a nonvanishing shear part of the stress-energy. As a by-product, we obtain the Bogolubov transformation for massive Dirac fermions on (ordinary) Rindler space in $(3+1)$ dimensions, which to the knowledge of the author has not appeared in the literature.¹

The rest of the paper is as follows. Massive fermions on the topologically nontrivial Rindler spaces are analysed in section 2, and the massless limit stress-energy is computed in section 3. Massive fermions on the \mathbb{RP}^3 geon are analysed in section 4. Section 5 gives a brief summary and discussion.

¹Thermality for massive fermions on $(3+1)$ -dimensional Rindler space is demonstrated by other methods in [14, 15]. The massive $(1+1)$ -dimensional case is considered in [16]. The massive $(3+1)$ case is addressed in [17] but the Rindler modes constructed therein are not suitably orthonormal in the Dirac inner product.

We work throughout in natural units, $\hbar = c = G = 1$. The metric signature is $(+, -, -, -)$. Complex conjugation is denoted by $*$ and charge conjugation by c .

2 The Unruh effect on M_0 and M_- for massive spinors

In this section we discuss the Unruh effect for the massive Dirac field on two flat spacetimes whose global properties mimic respectively those of the Kruskal manifold and the \mathbb{RP}^3 geon. In section 2.1 we recall the construction of these spacetimes, denoted by M_0 and M_- [10], and discuss the spin structures they admit. The Minkowski-like vacua on M_0 and M_- are constructed in section 2.2, and the Rindler-particle content of these vacua is found from the explicit Bogolubov transformation in sections 2.3 and 2.4.

2.1 The spacetimes M_0 and M_-

Let M denote $(3+1)$ -dimensional Minkowski space and let (t, x, y, z) be a standard set of Minkowski coordinates. The metric reads

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2. \quad (2.1)$$

The spacetimes M_0 and M_- are defined as quotients of M under the isometry groups generated respectively by the isometries

$$J_0 : (t, x, y, z) \mapsto (t, x, y, z + 2a), \quad (2.2a)$$

$$J_- : (t, x, y, z) \mapsto (t, -x, -y, z + a), \quad (2.2b)$$

where $a > 0$ is a prescribed constant. M_0 and M_- are space and time orientable flat Lorentzian manifolds, globally hyperbolic with spatial topology $\mathbb{R}^2 \times S^1$, and M_0 is a double cover of M_- . We may understand M_0 and M_- to be coordinatised by the Minkowski coordinates with the identifications

$$M_0 : (t, x, y, z) \sim (t, x, y, z + 2a), \quad (2.3a)$$

$$M_- : (t, x, y, z) \sim (t, -x, -y, z + a). \quad (2.3b)$$

For further discussion, see [10].

Due to the S^1 factor in the spatial topology, M_0 and M_- each admit two inequivalent spin structures (see e.g. [18]). We need a practical way to describe these spin structures.

Consider first M_0 . We refer to the vierbein

$$\begin{aligned} V_0 &= \partial_t & V_1 &= \partial_x \\ V_2 &= \partial_y & V_3 &= \partial_z \end{aligned} \quad (2.4)$$

as the standard vierbein on M_0 . In the standard vierbein, the two spin structures amount to imposing respectively periodic and antiperiodic boundary conditions on the spinors in the z -direction: Labelling the spin structures by the index $\epsilon \in \{1, -1\}$, this means

$$\psi(t, x, y, z + 2na) = \epsilon^n \psi(t, x, y, z), \quad (2.5)$$

where $n \in \mathbb{Z}$ and $\epsilon = +1$ for the periodic spinors and $\epsilon = -1$ for the antiperiodic spinors.

An alternative useful vierbein on M_0 is

$$\begin{aligned} V_0 &= \partial_t \\ V_1 &= \cos(\pi z/a) \partial_x + \sin(\pi z/a) \partial_y \\ V_2 &= -\sin(\pi z/a) \partial_x + \cos(\pi z/a) \partial_y \\ V_3 &= \partial_z \ , \end{aligned} \tag{2.6}$$

which rotates counterclockwise by 2π in the (x, y) plane as z increases by $2a$. Spinors that are periodic in the standard vierbein (2.4) are antiperiodic when written in the rotating vierbein (2.6) and vice versa. One could further introduce a vierbein that rotates clockwise by 2π in the (x, y) plane as z increases by $2a$ [(replace π with $-\pi$ in (2.6)], but periodic (respectively antiperiodic) boundary conditions in this vierbein are equivalent to periodic (antiperiodic) boundary conditions in vierbein (2.6). This shows that neither spin structure on M_0 involves a preferred spatial orientation.

Consider then M_- . The standard vierbein (2.4) is not invariant under J_- and does therefore not provide a globally-defined vierbein on M_- . However, both the counterclockwise-rotating vierbein (2.6) and its clockwise-rotating analogue are invariant under J_- and hence well defined on M_- . We may therefore specify the two spin structures on M_- by working in the vierbein (2.6) and imposing respectively periodic and antiperiodic boundary conditions under J_- , or equivalently working in the clockwise-rotating vierbein and interchanging the periodic and antiperiodic boundary conditions. This shows that the choice of a spin structure on M_- determines a preferred spatial orientation. For concreteness, we shall specify the spin structure with respect to the vierbein (2.6).

2.2 Minkowski-like vacua

In this subsection we quantise a free Dirac field ψ with mass $m \geq 0$ on M_0 and M_- , constructing Minkowski-like vacua defined with respect to the global timelike Killing vector ∂_t .

In a general curved spacetime the spinor Lagrangian density is given in the vierbein formalism by [7]

$$\mathcal{L} = \det V \left(\frac{1}{2} i [\bar{\psi} \gamma^\mu \nabla_\mu \psi - (\nabla_\mu \bar{\psi}) \gamma^\mu \psi] - m \bar{\psi} \psi \right) , \tag{2.7}$$

where V_α^μ is a vierbein, $V_\alpha = V_\alpha^\mu \partial_\mu$, γ^α are the flat space Dirac matrices, and $\gamma^\mu = V_\alpha^\mu \gamma^\alpha$ are the curved space Dirac matrices satisfying $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. The spinor covariant derivative is $\nabla_\alpha = V_\alpha^\mu (\partial_\mu + \Gamma_\mu)$ with $\Gamma_\mu = \frac{1}{8} V_\alpha^\nu V_{\beta\nu;\mu} [\gamma^\alpha, \gamma^\beta]$. Variation of the action yields the covariant Dirac equation

$$i \gamma^\mu \nabla_\mu \psi - m \psi = 0 . \tag{2.8}$$

It will be useful to work in the local Minkowski coordinates (t, x, y, z) and in the rotating vierbein (2.6), which is well-defined on both M_0 and M_- . The Dirac equation (2.8) then becomes

$$\begin{aligned} i \{ \gamma^0 \partial_t + \gamma^1 [\cos(\pi z/a) \partial_x + \sin(\pi z/a) \partial_y] + \gamma^2 [-\sin(\pi z/a) \partial_x + \cos(\pi z/a) \partial_y] \\ + \gamma^3 [\partial_z - \frac{1}{4}(\pi/a)(\gamma^1 \gamma^2 - \gamma^2 \gamma^1)] + im \} \psi = 0 . \end{aligned} \tag{2.9}$$

The inner product is

$$\langle \psi_1, \psi_2 \rangle = \int_{t=\text{const}} dx dy dz \psi_1^\dagger \psi_2 . \quad (2.10)$$

We denote the inner products on M_0 and M_- by $\langle \psi_1, \psi_2 \rangle_0$ and $\langle \psi_1, \psi_2 \rangle_-$ respectively.

Consider first M_0 . To construct solutions to (2.9) that are positive and negative frequency with respect to ∂_t , we begin with the Minkowski space positive and negative frequency solutions in the standard vierbein (2.4) (see for example [19, 20]) and transform to the rotating vierbein (2.6) by the spinor transformation associated with a rotation by π in the (x, y) plane as z increases by a . Working here and throughout this section in the standard representation of the γ matrices [20], $\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$, where σ_i are the Pauli matrices, this transformation reads

$$\psi \mapsto e^{-\frac{\gamma^1 \gamma^2 \pi z}{2a}} \psi = \text{diag} \left(e^{\frac{i\pi z}{2a}}, e^{-\frac{i\pi z}{2a}}, e^{\frac{i\pi z}{2a}}, e^{-\frac{i\pi z}{2a}} \right) \psi . \quad (2.11)$$

The periodic and antiperiodic boundary conditions will then restrict the momentum in the z -direction. We find that a complete set of normalised positive frequency solutions is $\{U_{j,k_x,k_y,k_z}\}$, where

$$U_{j,k_x,k_y,k_z} = \frac{1}{4\pi} \sqrt{\frac{(\omega+m)}{a\omega}} e^{-i\omega t + ik_x x + ik_y y + ik_z z} u_{j,k_x,k_y,k_z} , \quad (2.12)$$

with

$$u_1 = \begin{pmatrix} e^{\frac{i\pi z}{2a}} \\ 0 \\ e^{\frac{i\pi z}{2a}} \frac{k_z}{\omega+m} \\ e^{-\frac{i\pi z}{2a}} \frac{k_+}{\omega+m} \end{pmatrix} , \quad u_2 = \begin{pmatrix} 0 \\ e^{-\frac{i\pi z}{2a}} \\ e^{\frac{i\pi z}{2a}} \frac{k_-}{\omega+m} \\ e^{-\frac{i\pi z}{2a}} \frac{-k_z}{\omega+m} \end{pmatrix} , \quad (2.13)$$

$k_\pm = k_x \pm ik_y$ and $\omega = \sqrt{m^2 + k_x^2 + k_y^2 + k_z^2}$. For spinors that are periodic in the standard vierbein (2.4), $k_z = n\pi/a$ with $n \in \mathbb{Z}$, and in the other spin structure $k_z = (n + \frac{1}{2})\pi/a$ with $n \in \mathbb{Z}$. k_x and k_y take all real values. The orthonormality relation is

$$\langle U_{i,k_x,k_y,k_z}, U_{j,k'_x,k'_y,k'_z} \rangle_0 = \delta_{ij} \delta_{nn'} \delta(k_x - k'_x) \delta(k_y - k'_y) . \quad (2.14)$$

Note that the corresponding delta-normalised modes on Minkowski are $\sqrt{\frac{a}{\pi}} U_{j,k_x,k_y,k_z}$ with $k_z \in \mathbb{R}$.

Consider then M_- . As M_0 is a double cover of M_- , a complete set of positive frequency modes is obtained by superposing the modes (2.12) and their images under J_- with phase factors that lead to the appropriate (anti-)periodicity properties. We choose the set $\{V_{j,k_x,k_y,k_z}\}$ given by

$$\begin{aligned} V_{1,k_x,k_y,k_z} &= U_{1,k_x,k_y,k_z} + \epsilon i e^{ik_z a} U_{1,-k_x,-k_y,k_z} , \\ V_{2,k_x,k_y,k_z} &= U_{2,k_x,k_y,k_z} - \epsilon i e^{ik_z a} U_{2,-k_x,-k_y,k_z} , \end{aligned} \quad (2.15)$$

where $k_z = (n + \frac{1}{2})\pi/a$, $n \in \mathbb{Z}$ and $\epsilon = 1$ ($\epsilon = -1$) gives spinors that are periodic (antiperiodic) in the rotating vierbein (2.6). As with the scalar field [10] there is a

redundancy in these V -modes in that V_{j,k_x,k_y,k_z} and $V_{j,-k_x,-k_y,k_z}$ are proportional, and we understand this redundancy to be eliminated by taking for example $k_y > 0$. The orthonormality condition reads

$$\langle V_{i,k_x,k_y,k_z}, V_{j,k'_x,k'_y,k'_z} \rangle_- = \delta_{ij} \delta_{nn'} \delta(k_x - k'_x) \delta(k_y - k'_y) . \quad (2.16)$$

Note that k_z takes in both spin structures the same set of values, which coincides with the set in the twisted spin structure on M_0 .

Given these mode sets, we can canonically quantise in the usual way, expanding the field in the modes and imposing the usual anticommutation relations on the coefficients. Let $|0\rangle$ be the usual Minkowski vacuum on M , defined by the set $\{U_{j,k_x,k_y,k_z}\}$. We denote by $|0_0\rangle$ the vacuum on M_0 defined by the set $\{U_{j,k_x,k_y,k_z}\}$ with the suitably restricted values of k_z and by $|0_-\rangle$ the vacuum on M_- defined by the set $\{V_{j,k_x,k_y,k_z}\}$. $|0_0\rangle$ and $|0_-\rangle$ both depend on the respective spin structures, but in what follows we will not need an explicit index to indicate this dependence.

2.3 Bogolubov transformation on M_0

In this subsection we find the Rindler-particle content of the Minkowski-like vacuum on M_0 from the explicit Bogolubov transformation. At the end of the subsection we indicate how the corresponding results for Minkowski space can be read off from our formulas.

Let R_0 be the right-hand-side Rindler wedge of M_0 , $x > |t|$. We introduce in R_0 the usual Rindler coordinates (η, ρ, y, z) by

$$\begin{aligned} t &= \rho \sinh \eta , \\ x &= \rho \cosh \eta , \end{aligned} \quad (2.17)$$

with $\rho > 0$ and $-\infty < \eta < \infty$, understood with the identification $(\eta, \rho, y, z) \sim (\eta, \rho, y, z + 2a)$. The metric reads

$$ds^2 = \rho^2 d\eta^2 - d\rho^2 - dy^2 - dz^2 . \quad (2.18)$$

R_0 is a globally hyperbolic spacetime with the complete timelike Killing vector $\partial_\eta = t\partial_x + x\partial_t$, which generates boosts in the (t, x) plane. The worldlines at constant ρ , y and z are those of observers accelerated uniformly in the x -direction with acceleration ρ^{-1} and proper time $\rho\eta$.

We need in R_0 a set of orthonormal field modes that are positive frequency with respect to ∂_η . In the vierbein aligned along the Rindler coordinate axes,

$$V_a^\mu = \text{diag}(\rho^{-1}, 1, 1, 1) , \quad (2.19)$$

the Dirac equation (2.8) becomes

$$(i\partial_\eta + i\rho\gamma^0\gamma^1\partial_\rho + i\rho\gamma^0\gamma^2\partial_y + i\rho\gamma^0\gamma^3\partial_z + i\gamma^0\gamma^1/2 - m\rho\gamma^0)\psi = 0 , \quad (2.20)$$

where the γ matrices are the usual flat space γ 's. We separate (2.20) by an ansatz of simultaneous eigenfunctions of $-i\partial_y$, $-i\partial_z$ and the Rindler Hamiltonian. In view

of comparison with M_- in subsection 2.4, we wish the solutions to have simple transformation properties under J_- . Modes that achieve this are

$$\psi_{j,M,k_y,k_z}^R(\eta, \rho, y, z) = N_j \left(X_j^R K_{iM-\frac{1}{2}}(k\rho) + Y_j^R K_{iM+\frac{1}{2}}(k\rho) \right) e^{-iM\eta + ik_y y + ik_z z}, \quad (2.21)$$

where

$$\begin{aligned} X_1^R &= \begin{pmatrix} \frac{k_z}{|k_z|}(k_y - im) \\ -i(|k_z| - \kappa) \\ -i(|k_z| - \kappa) \\ \frac{k_z}{|k_z|}(k_y - im) \end{pmatrix}, & Y_1^R &= \begin{pmatrix} \frac{k_z}{|k_z|}(|k_z| - \kappa) \\ i(k_y - im) \\ -i(k_y - im) \\ -\frac{k_z}{|k_z|}(|k_z| - \kappa) \end{pmatrix}, \\ X_2^R &= \begin{pmatrix} \frac{k_z}{|k_z|}(|k_z| - \kappa) \\ i(k_y + im) \\ i(k_y + im) \\ \frac{k_z}{|k_z|}(|k_z| - \kappa) \end{pmatrix}, & Y_2^R &= \begin{pmatrix} \frac{k_z}{|k_z|}(k_y + im) \\ -i(|k_z| - \kappa) \\ i(|k_z| - \kappa) \\ -\frac{k_z}{|k_z|}(k_y + im) \end{pmatrix}, \end{aligned} \quad (2.22)$$

and

$$\begin{aligned} N_1 &= \frac{e^{-\frac{i\pi}{4}} \sqrt{\cosh(\pi M)(\kappa^2 - k_z^2)}}{4\pi(k_y - im) \sqrt{a\pi(\kappa - |k_z|)}}, \\ N_2 &= \frac{e^{-\frac{i\pi}{4}} \sqrt{\cosh(\pi M)(\kappa^2 - k_z^2)}}{4\pi(k_y + im) \sqrt{a\pi(\kappa - |k_z|)}}, \end{aligned} \quad (2.23)$$

$j = 1, 2$, $\kappa = (m^2 + k_y^2 + k_z^2)^{1/2}$, $M > 0$ and $k_y \in \mathbb{R}$. In the spin structure where the spinors are periodic (respectively antiperiodic) in the nonrotating vierbein (2.19), the values of k_z are $n\pi/a$ (respectively $(n + \frac{1}{2})\pi/a$) with $n \in \mathbb{Z}$. $K_{iM+\frac{1}{2}}$ is a modified Bessel function [21]. For $k_z = 0$, we understand the formulas in (2.21)–(2.23) and in what follows to stand for their limiting values as $k_z \rightarrow 0_+$. The modes are orthonormal as

$$\langle \psi_{i,M,k_y,k_z}^R, \psi_{j,M',k'_y,k'_z}^R \rangle_{R_0} = \delta_{ij} \delta_{nn'} \delta(M - M') \delta(k_y - k'_y), \quad (2.24)$$

where the inner product is (see e.g. ([15]))

$$\langle \psi_1, \psi_2 \rangle_{R_0} = \int d\rho dy dz \psi_1^\dagger \psi_2, \quad (2.25)$$

taken on an $\eta = \text{constant}$ hypersurface.

While the above modes would be sufficient for quantising in R_0 in its own right, they are not suitable for analytic continuation arguments across the horizons, as the vierbein (2.19) becomes singular in the limit $x \rightarrow |t|$. We therefore express the modes in the vierbein (2.6), which is globally defined on M_0 . This vierbein will further make the comparison to M_- transparent in subsection 2.4. The Lorentz transformation between (2.19) and (2.6) is a boost by rapidity $-\eta$ in the (η, ρ) plane followed by a rotation by π as $z \mapsto z + a$ in the (x, y) plane. The corresponding transformation on the spinors is

$$\psi \mapsto e^{-\frac{\gamma^1 \gamma^2 \pi z}{2a}} e^{\frac{\gamma^0 \gamma^1 \eta}{2}} \psi. \quad (2.26)$$

In the vierbein (2.6), our solutions thus become

$$\begin{aligned} \psi_{j,M,k_y,k_z}^R(t,x,y,z) = N_j \Big(X_j'^R K_{iM-\frac{1}{2}}(k\rho) e^{-(iM-\frac{1}{2})\eta} \\ + Y_j'^R K_{iM+\frac{1}{2}}(k\rho) e^{-(iM+\frac{1}{2})\eta} \Big) e^{ik_y y + ik_z z}, \end{aligned} \quad (2.27)$$

where

$$\begin{aligned} X_1'^R &= \begin{pmatrix} e^{\frac{i\pi z}{2a} \frac{k_z}{|k_z|}} (k_y - im) \\ -e^{-\frac{i\pi z}{2a}} i(|k_z| - \kappa) \\ -e^{\frac{i\pi z}{2a}} i(|k_z| - \kappa) \\ e^{-\frac{i\pi z}{2a} \frac{k_z}{|k_z|}} (k_y - im) \end{pmatrix}, & Y_1'^R &= \begin{pmatrix} e^{\frac{i\pi z}{2a} \frac{k_z}{|k_z|}} (|k_z| - \kappa) \\ e^{-\frac{i\pi z}{2a}} i(k_y - im) \\ -e^{\frac{i\pi z}{2a}} i(k_y - im) \\ -e^{-\frac{i\pi z}{2a} \frac{k_z}{|k_z|}} (|k_z| - \kappa) \end{pmatrix}, \\ X_2'^R &= \begin{pmatrix} e^{\frac{i\pi z}{2a} \frac{k_z}{|k_z|}} (|k_z| - \kappa) \\ e^{-\frac{i\pi z}{2a}} i(k_y + im) \\ e^{\frac{i\pi z}{2a}} i(k_y + im) \\ e^{-\frac{i\pi z}{2a} \frac{k_z}{|k_z|}} (|k_z| - \kappa) \end{pmatrix}, & Y_2'^R &= \begin{pmatrix} e^{\frac{i\pi z}{2a} \frac{k_z}{|k_z|}} (k_y + im) \\ -e^{-\frac{i\pi z}{2a}} i(|k_z| - \kappa) \\ e^{\frac{i\pi z}{2a}} i(|k_z| - \kappa) \\ -e^{-\frac{i\pi z}{2a} \frac{k_z}{|k_z|}} (k_y + im) \end{pmatrix}. \end{aligned} \quad (2.28)$$

We proceed similarly in the left-hand-side Rindler wedge L_0 , $x < -|t|$. We define the Rindler coordinates in L_0 by

$$\begin{aligned} t &= -\rho \sinh \eta, \\ x &= -\rho \cosh \eta, \end{aligned} \quad (2.29)$$

again with $\rho > 0$ and $-\infty < \eta < \infty$. Note that ∂_η is now past-pointing. In the vierbein (2.6), a complete orthonormal set of positive frequency modes with respect to ∂_η is

$$\begin{aligned} \psi_{j,M,k_y,k_z}^L(t,x,y,z) = N_j \Big(X_j'^L K_{iM-\frac{1}{2}}(k\rho) e^{-(iM-\frac{1}{2})\eta} \\ + Y_j'^L K_{iM+\frac{1}{2}}(k\rho) e^{-(iM+\frac{1}{2})\eta} \Big) e^{ik_y y + ik_z z}, \end{aligned} \quad (2.30)$$

where

$$\begin{aligned} X_1'^L &= \begin{pmatrix} e^{\frac{i\pi z}{2a} \frac{k_z}{|k_z|}} (k_y - im) \\ -e^{-\frac{i\pi z}{2a}} i(|k_z| - \kappa) \\ -e^{\frac{i\pi z}{2a}} i(|k_z| - \kappa) \\ e^{-\frac{i\pi z}{2a} \frac{k_z}{|k_z|}} (k_y - im) \end{pmatrix}, & Y_1'^L &= \begin{pmatrix} -e^{\frac{i\pi z}{2a} \frac{k_z}{|k_z|}} (|k_z| - \kappa) \\ -e^{-\frac{i\pi z}{2a}} i(k_y - im) \\ e^{\frac{i\pi z}{2a}} i(k_y - im) \\ e^{-\frac{i\pi z}{2a} \frac{k_z}{|k_z|}} (|k_z| - \kappa) \end{pmatrix}, \\ X_2'^L &= \begin{pmatrix} -e^{\frac{i\pi z}{2a} \frac{k_z}{|k_z|}} (|k_z| - \kappa) \\ -e^{-\frac{i\pi z}{2a}} i(k_y + im) \\ -e^{\frac{i\pi z}{2a}} i(k_y + im) \\ -e^{-\frac{i\pi z}{2a} \frac{k_z}{|k_z|}} (|k_z| - \kappa) \end{pmatrix}, & Y_2'^L &= \begin{pmatrix} e^{\frac{i\pi z}{2a} \frac{k_z}{|k_z|}} (k_y + im) \\ -e^{-\frac{i\pi z}{2a}} i(|k_z| - \kappa) \\ e^{\frac{i\pi z}{2a}} i(|k_z| - \kappa) \\ -e^{-\frac{i\pi z}{2a} \frac{k_z}{|k_z|}} (k_y + im) \end{pmatrix}, \end{aligned} \quad (2.31)$$

and the ranges of M , k_y and k_z are as in R_0 . The orthonormality relation is similar to (2.25).

We may now quantise the field in R_0 and L_0 in the usual manner. A complete set of positive frequency modes with respect to the future-pointing timelike Killing

vector is $\{\psi_{j,M,k_y,k_z}^R(t,x,y,z)\}$ in R_0 and $\{\psi_{j,-M,k_y,k_z}^L(t,x,y,z)\}$ in L_0 , both with $M > 0$: The minus sign in ψ^L arises because ∂_η is past-pointing in L_0 . The expansion of the field in these modes and their charge conjugates reads

$$\begin{aligned} \Psi = \sum_j \sum_n \int_0^\infty dM \int_{-\infty}^\infty dk_y \Big(& a_{j,M,k_y,k_z}^R \psi_{j,M,k_y,k_z}^R + a_{j,M,k_y,k_z}^L \psi_{j,-M,k_y,k_z}^L \\ & + b_{j,M,k_y,k_z}^{R\dagger} \psi_{j,M,k_y,k_z}^{R,c} + b_{j,M,k_y,k_z}^{L\dagger} \psi_{j,-M,k_y,k_z}^{L,c} \Big) . \end{aligned} \quad (2.32)$$

As the charge conjugation in our standard representation of the γ 's reads $\psi^c = i\gamma^2\psi^*$, where the superscript $*$ stands for normal complex conjugation, it follows that

$$\begin{aligned} \psi_{1,M,k_y,k_z}^{R,c}(t,x,y,z) &= -i\psi_{1,-M,-k_y,-k_z}^R(t,x,y,z) , \\ \psi_{2,M,k_y,k_z}^{R,c}(t,x,y,z) &= i\psi_{2,-M,-k_y,-k_z}^R(t,x,y,z) , \end{aligned} \quad (2.33)$$

with similar expressions for ψ_1^L and ψ_2^L . The annihilation and creation operators in R_0 satisfy the usual anticommutation relations

$$\begin{aligned} \{a_{i,M,k_y,k_z}^R, a_{j,M',k'_y,k'_z}^{R\dagger}\} &= \delta_{ij}\delta_{nn'}\delta(M-M')\delta(k_y-k'_y) , \\ \{b_{i,M,k_y,k_z}^R, b_{j,M',k'_y,k'_z}^{R\dagger}\} &= \delta_{ij}\delta_{nn'}\delta(M-M')\delta(k_y-k'_y) , \end{aligned} \quad (2.34)$$

with similar relations holding in L_0 and all mixed anticommutators vanishing. The right and left Rindler vacua, $|0_{R_0}\rangle$ and $|0_{L_0}\rangle$, are defined as the states annihilated by all the appropriate annihilation operators.

Now, we wish to find the Rindler mode content of the Minkowski-like vacuum $|0_0\rangle$ by Unruh's analytic continuation method [5]. To begin, we continue the sets $\{\psi_{j,M,k_y,k_z}^R\}$ and $\{\psi_{j,-M,k_y,k_z}^L\}$ analytically to all of M_0 , crossing the horizons in the lower half-plane in complexified t . Using the relations (2.33), their counterparts in L_0 and the complex analytic properties of the Bessel functions [21], we find that the resulting modes are

$$\begin{aligned} W_{j,M,k_y,k_z}^{(1)}(t,x,y,z) &= \frac{1}{\sqrt{2\cosh(\pi M)}} \left(e^{\frac{\pi M}{2}} \psi_{j,M,k_y,k_z}^R + e^{-\frac{\pi M}{2}} \psi_{j,-M,-k_y,-k_z}^{L,c} \right) , \\ W_{j,M,k_y,k_z}^{(2)}(t,x,y,z) &= \frac{1}{\sqrt{2\cosh(\pi M)}} \left(-e^{-\frac{\pi M}{2}} \psi_{j,M,-k_y,-k_z}^{R,c} + e^{\frac{\pi M}{2}} \psi_{j,-M,k_y,k_z}^L \right) , \end{aligned} \quad (2.35)$$

where $M > 0$. The normalisation is

$$\langle W_{i,M,k_y,k_z}, W_{j,M',k'_y,k'_z} \rangle_0 = \delta_{ij}\delta_{nn'}\delta(M-M')\delta(k_y-k'_y) . \quad (2.36)$$

We then expand the field as

$$\begin{aligned} \Psi = \sum_j \sum_n \int_0^\infty dM \int_{-\infty}^\infty dk_y \Big(& c_{j,M,k_y,k_z}^{(1)} W_{j,M,k_y,k_z}^{(1)} + c_{j,M,k_y,k_z}^{(2)} W_{j,M,k_y,k_z}^{(2)} \\ & + d_{j,M,k_y,k_z}^{(1)\dagger} W_{j,M,k_y,k_z}^{(1),c} + d_{j,M,k_y,k_z}^{(2)\dagger} W_{j,M,k_y,k_z}^{(2),c} \Big) \end{aligned} \quad (2.37)$$

and impose the usual anticommutation relations for the creation and annihilation operators in (2.37). Equating (2.37) and (2.32) and taking inner products with the ψ modes gives the Bogolubov transformation

$$\begin{aligned} a_{j,M,k_y,k_z}^R &= \frac{1}{\sqrt{2 \cosh(\pi M)}} \left(e^{\frac{\pi M}{2}} c_{j,M,k_y,k_z}^{(1)} - e^{-\frac{\pi M}{2}} d_{j,M,-k_y,-k_z}^{(2)\dagger} \right) , \\ a_{j,M,k_y,k_z}^L &= \frac{1}{\sqrt{2 \cosh(\pi M)}} \left(e^{\frac{\pi M}{2}} c_{j,M,k_y,k_z}^{(2)} + e^{-\frac{\pi M}{2}} d_{j,M,-k_y,-k_z}^{(1)\dagger} \right) , \\ b_{j,M,k_y,k_z}^{R,\dagger} &= \frac{1}{\sqrt{2 \cosh(\pi M)}} \left(e^{\frac{\pi M}{2}} d_{j,M,k_y,k_z}^{(1)\dagger} - e^{-\frac{\pi M}{2}} c_{j,M,-k_y,-k_z}^{(2)} \right) , \\ b_{j,M,k_y,k_z}^{L,\dagger} &= \frac{1}{\sqrt{2 \cosh(\pi M)}} \left(e^{\frac{\pi M}{2}} d_{j,M,k_y,k_z}^{(2)\dagger} + e^{-\frac{\pi M}{2}} c_{j,M,-k_y,-k_z}^{(1)} \right) . \end{aligned} \quad (2.38)$$

As the W -modes are by construction purely positive frequency with respect to ∂_t [5], the vacuum annihilated by all the annihilation operators in (2.37) is $|0_0\rangle$. This observation and the transformation (2.38) fix the coefficients in the expansion of $|0_0\rangle$ in terms of the Rindler-excitations on $|0_{R_0}\rangle$ and $|0_{L_0}\rangle$. The final result is

$$|0_0\rangle = \prod_{j,M,k_y,k_z} \frac{1}{(e^{-2\pi M} + 1)^{\frac{1}{2}}} \sum_{q=0,1} (-1)^q e^{-\pi M q} |q\rangle_{j,M,k_y,k_z}^R |q\rangle_{j,M,-k_y,-k_z}^L , \quad (2.39)$$

where the notation on the right hand side is adapted to the tensor product structure of the Hilbert space considered:

$$\begin{aligned} |q\rangle_{j,M,k_y,k_z}^R &= (a_{j,M,k_y,k_z}^{R\dagger})^q |0_{R_0}\rangle , \\ |q\rangle_{j,M,k_y,k_z}^L &= (b_{j,M,k_y,k_z}^{L\dagger})^q |0_{L_0}\rangle . \end{aligned} \quad (2.40)$$

The result (2.39) is the massive fermion version of the familiar bosonic result [7], indicating an entangled state in which the right and left Rindler excitations appear in correlated pairs. An operator $\hat{A}^{(1)}$ whose support is in R_0 does not couple to the Rindler-modes in L_0 and has hence the expectation value $\langle 0_0 | \hat{A}^{(1)} | 0_0 \rangle = \text{Tr}(\hat{A}^{(1)} \rho^{(1)})$, where $\rho^{(1)}$ is a fermionic thermal density matrix in R_0 ,

$$\rho^{(1)} = \prod_{j,M,k_y,k_z} \sum_{q=0,1} \frac{e^{-2q\pi M}}{\sum_{m=0,1} e^{-2m\pi M}} |q\rangle_{j,M,k_y,k_z}^R \langle q|_{j,M,k_y,k_z}^R . \quad (2.41)$$

In particular, the number operator expectation value takes the fermionic thermal form,

$$\langle 0_0 | a_{i,M,k_y,k_z}^{R\dagger} a_{j,M',k_y',k_z'}^R | 0_0 \rangle = \frac{1}{(e^{2\pi M} + 1)} \delta_{ij} \delta_{nn'} \delta(k_y - k_y') \delta(M - M') . \quad (2.42)$$

The delta-function divergences in (2.42) arise from the continuum normalisation of our modes and can be remedied by wave packets as in the scalar case [10]. This shows that the Rindler-observers in R_0 see $|0_0\rangle$ as a thermal bath at the usual Unruh temperature, $T = (2\pi\rho)^{-1}$. Similar considerations clearly hold for L_0 .

The result (2.39) incorporates the two spin structures on M_0 , R_0 and L_0 , in the allowed values of k_z , and these values are the same in all our mode sets. The twisted

(respectively untwisted) $|0_0\rangle$ thus induces twisted (untwisted) thermal states in both R_0 and L_0 .

To end this subsection, we note that the Bogolubov transformation on Minkowski space can be read off from our formulas on M_0 with minor systematic changes. There is now only one spin structure and k_z takes all real values. The expressions for the various mode functions include the additional factor $\sqrt{a/\pi}$, sums over n become integrals over k_z , and in the normalisation and anticommutation relations the discrete delta $\delta_{nn'}$ is replaced by the delta-function $\delta(k_z - k'_z)$. The formulas involve still a because the spinors are expressed in the rotating vierbein (2.6). Translation into the standard vierbein (2.4) can be accomplished by the appropriate spinor transformation.

2.4 Bogolubov transformation on M_-

In this subsection we find the Rindler-particle content of the Minkowski-like vacuum on M_- .

Let R_- denote the Rindler wedge on M_- , given in our local coordinates by $x > |t|$. As R_- is isometric to R_0 , we may introduce in R_- the Rindler-coordinates (η, ρ, y, z) by (2.17), again with the identifications $(\eta, \rho, y, z) \sim (\eta, \rho, y, z + 2a)$, and quantise as in R_0 , defining the positive and negative frequencies with respect to the Killing vector ∂_η . For convenience of phase factors in the W -modes (2.45) below, we use the mode set $\{\Psi_{j,M,k_y,k_z}^R\}$, defined as in (2.27) except in that the normalisation factors (2.23) are replaced by

$$\begin{aligned} N_1 &= \frac{e^{-\frac{ik_z a}{2}} \sqrt{\cosh(\pi M)(\kappa^2 - k_z^2)}}{4\pi(k_y - im) \sqrt{a\pi(\kappa - |k_z|)}} , \\ N_2 &= \frac{e^{\frac{ik_z a}{2}} \sqrt{\cosh(\pi M)(\kappa^2 - k_z^2)}}{4\pi(k_y + im) \sqrt{a\pi(\kappa - |k_z|)}} . \end{aligned} \quad (2.43)$$

A key difference from R_0 arises, however, from the requirement that the spinors on R_- must be extendible into spinors in one of the two spin structures on M_- . By the discussion in subsection 2.1, this implies that k_z is restricted to the values $k_z = (n + \frac{1}{2})\pi/a$ with $n \in \mathbb{Z}$. Both of the spin structures on M_- thus induce on R_- the same spin structure, in which the spinors are antiperiodic in the nonrotating vierbein (2.4).

The inner product on R_- is as in (2.25) and the orthonormality relation is similar to (2.24). The mode expansion of the field reads

$$\begin{aligned} \Psi = \sum_n \int_0^\infty dM \int_{-\infty}^\infty dk_y & \left(a_{1,M,k_y,k_z} \Psi_{1,M,k_y,k_z}^R + a_{2,M,k_y,k_z} \Psi_{2,M,k_y,k_z}^R \right. \\ & \left. + b_{1,M,k_y,k_z}^\dagger \Psi_{1,M,k_y,k_z}^{R,c} + b_{2,M,k_y,k_z}^\dagger \Psi_{2,M,k_y,k_z}^{R,c} \right) , \end{aligned} \quad (2.44)$$

where the annihilation and creation operators satisfy the usual anticommutation relations. The Rindler-vacuum $|0_{R_-}\rangle$ on R_- is the state annihilated by all the annihilation operators in (2.44).

To find the Rindler-mode content of $|0_-\rangle$, we again use the analytic continuation method. Working in the local coordinates (t, x, y, z) , we continue the modes

$\{\Psi_{j,M,k_y,k_z}^R\}$ across the horizons in the lower half-plane in complexified t and form the linear combinations that are globally well-defined on M_- . The complete set of the resulting W -modes is $\{W_{j,M,k_y,k_z}^R\}$, given by

$$\begin{aligned} W_{1,M,k_y,k_z}^R(t, x, y, z) &= \frac{1}{\sqrt{2 \cosh \pi M}} \left(e^{\frac{\pi M}{2}} \Psi_{1,M,k_y,k_z}^R + \epsilon e^{-\frac{\pi M}{2}} \Psi_{2,M,k_y,-k_z}^{R,c} \right) , \\ W_{2,M,k_y,k_z}^R(t, x, y, z) &= \frac{1}{\sqrt{2 \cosh \pi M}} \left(e^{\frac{\pi M}{2}} \Psi_{2,M,k_y,k_z}^R - \epsilon e^{-\frac{\pi M}{2}} \Psi_{1,M,k_y,-k_z}^{R,c} \right) , \end{aligned} \quad (2.45)$$

where $M > 0$. The orthonormality relation is similar to (2.24). We have introduced the parameter ϵ in (2.45) to label the spin structure: $\epsilon = 1$ (respectively -1) gives spinors that are periodic (antiperiodic) in the globally-defined vierbein (2.6).

The expansion of the field in the W -modes reads

$$\begin{aligned} \Psi = \sum_n \int_0^\infty dM \int_{-\infty}^\infty dk_y &\left(c_{1,M,k_y,k_z} W_{1,M,k_y,k_z}^R + c_{2,M,k_y,k_z} W_{2,M,k_y,k_z}^R \right. \\ &\left. + d_{1,M,k_y,k_z}^\dagger W_{1,M,k_y,k_z}^{R,c} + d_{2,M,k_y,k_z}^\dagger W_{2,M,k_y,k_z}^{R,c} \right) , \end{aligned} \quad (2.46)$$

The annihilation and creation operators in (2.46) satisfy the usual anticommutation relations, and $|0_- \rangle$ is the state annihilated by the annihilation operators. It follows that the Bogolubov transformation between the Rindler-modes and the W -modes reads

$$\begin{aligned} a_{1,M,k_y,k_z} &= \frac{1}{\sqrt{2 \cosh(\pi M)}} \left(e^{\frac{\pi M}{2}} c_{1,M,k_y,k_z} - \epsilon e^{-\frac{\pi M}{2}} d_{2,M,k_y,-k_z}^\dagger \right) , \\ a_{2,M,k_y,k_z} &= \frac{1}{\sqrt{2 \cosh(\pi M)}} \left(e^{\frac{\pi M}{2}} c_{2,M,k_y,k_z} + \epsilon e^{-\frac{\pi M}{2}} d_{1,M,k_y,-k_z}^\dagger \right) , \\ b_{1,M,k_y,k_z}^\dagger &= \frac{1}{\sqrt{2 \cosh(\pi M)}} \left(e^{\frac{\pi M}{2}} d_{1,M,k_y,k_z}^\dagger - \epsilon e^{-\frac{\pi M}{2}} c_{2,M,k_y,-k_z} \right) , \\ b_{2,M,k_y,k_z}^\dagger &= \frac{1}{\sqrt{2 \cosh(\pi M)}} \left(e^{\frac{\pi M}{2}} d_{2,M,k_y,k_z}^\dagger + \epsilon e^{-\frac{\pi M}{2}} c_{1,M,k_y,-k_z} \right) , \end{aligned} \quad (2.47)$$

and the expansion of $|0_- \rangle$ in terms of the Rindler-excitations is

$$|0_- \rangle = \prod_{M,k_y,k_z} \frac{1}{(e^{-2\pi M} + 1)^{\frac{1}{2}}} \sum_{q=0,1} (-\epsilon)^q e^{-\pi M q} |q\rangle_{1,M,k_y,k_z} |q\rangle_{2,M,k_y,-k_z} , \quad (2.48)$$

where

$$\begin{aligned} |q\rangle_{1,M,k_y,k_z} &= a_{1,M,k_y,k_z}^\dagger |0_{R_-} \rangle , \\ |q\rangle_{2,M,k_y,k_z} &= b_{2,M,k_y,k_z}^\dagger |0_{R_-} \rangle , \end{aligned} \quad (2.49)$$

From (2.48) it is seen that $|0_- \rangle$ is an entangled state of Rindler-excitations, and the correlations are between a particle and an antiparticle with opposite eigenvalues of k_z . As all the excitations are in the unique Rindler wedge R_- , the expectation values of generic operators in R_- are not thermal. However, for any operator that only couples to one member of each correlated pair in R_- , the expectation values

are indistinguishable from those in the corresponding state $|0_0\rangle$ in R_0 , indicating thermality in the standard Unruh temperature. This is the case for example for any operator that only couples to excitations with a definite sign of k_z . In particular, the number operator expectation values are indistinguishable from (2.42), and it can be argued from the isometries as in [10] that the experiences of a Rindler-observer become asymptotically thermal at early or late times.

A key result of our analysis is that while both spin structures on M_- induce a state in the same spin structure in R_- , the explicit appearance of ϵ in (2.48) shows that the two states differ. A Rindler-observer in R_- can therefore in principle detect the spin structure on M_- from the nonthermal correlations. How these correlations could be detected in practice, for example by particle detectors with a local coupling to the field, presents an interesting question for future work. As the restriction of $|0_-\rangle$ to R_- is not invariant under the Killing vector ∂_η , and as the isometry arguments show that the correlations disappear in the limit of large $|\eta|$, investigating this question would require a particle detector formalism that can accommodate time dependent situations [22].

3 Stress-energy on M_0 and M_- for massless spinors

In this section we find the stress-energy expectation value for massless spinors in the Minkowski-like vacua on M_0 and M_- .

Starting with the spinors of section 2.2, we set $m = 0$ and adopt the Weyl (chiral) representation of the γ -matrices [20]. Writing the 4-component spinor as $\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$, the left-handed and right-handed 2-component spinors ψ_L and ψ_R then decouple, and it suffices to consider the stress-energy individually for each. This question was addressed by Banach and Dowker [23] in a class of spatially compact flat spacetimes that includes quotients of M_0 and M_- . As the quotients are handled by taking image sums, the stress-energy on M_0 and M_- is obtained from the results in [23] by dropping the summations that arise from the further quotients and matching the notation of [23] to ours.² The remaining sums are over a single integer: The sums in the diagonal components are purely numerical with well-known values [21], and the sums in the nondiagonal components on M_- can be evaluated by residues (see e.g. [24], Chapter 7). We summarise the results in Table 1.

The results in Table 1 show that the stress-energy does not distinguish right-handed and left-handed spinors, but it distinguishes M_0 from M_- and the two spin structures on each. On M_0 the $\epsilon = -1$ spin structure is energetically preferred, while on M_- both spin structures have the same energy density. On M_0 the stress-energy tensor is diagonal and invariant under all isometries. On M_- the stress-energy is invariant under all the continuous isometries, as it by construction must be, but there is now a nonzero shear component $\langle T_{\hat{\phi}z} \rangle$ whose sign is that of ϵ , and this sign changes under isometries that reverse the spatial orientation. On M_- , the spatial orientation determined by the spin structure can thus be detected from the shear part of the stress-energy. Note that the shear part vanishes at $r = 0$ and tends to

²There is a discrepancy in [23] in the definition of the matrix $S(x)$ between the Appendix and the bulk of the paper, one being the inverse of the other, and this affects the sign in the last factor in equation (60) therein and the related discussion at the top of page 2558. We agree however with the final stress-energy results in [23].

	$M_0, \epsilon = 1$	$M_0, \epsilon = -1$	M_-
$\langle T_{00} \rangle$	$\frac{\pi^2}{720a^4}$	$-\frac{7\pi^2}{5760a^4}$	$-\frac{7\pi^2}{5760a^4}$
$\langle T_{rr} \rangle = \langle T_{\hat{\phi}\hat{\phi}} \rangle$	$-\frac{\pi^2}{720a^4}$	$\frac{7\pi^2}{5760a^4}$	$\frac{7\pi^2}{5760a^4}$
$\langle T_{zz} \rangle$	$\frac{\pi^2}{240a^4}$	$-\frac{7\pi^2}{1920a^4}$	$-\frac{7\pi^2}{1920a^4}$
$\langle T_{\hat{\phi}z} \rangle$	0	0	$-\frac{\epsilon\pi^2}{64a^4} \frac{d}{dq} \left(\frac{\sinh q}{q \cosh^2 q} \right)$

Table 1: The nonvanishing components of $\langle T_{\mu\nu} \rangle$ for a left-handed two-component spinor in the Minkowski-like vacua on M_0 and M_- in the orthonormal frame $\{dt, dr, \omega^{\hat{\phi}}, dz\}$, where $x = r \cos \phi$, $y = r \sin \phi$ and $\omega^{\hat{\phi}} = r d\phi$, and $q = \pi(r/a)$. On M_0 , $\epsilon = 1$ (respectively -1) indicates spinors that are periodic (antiperiodic) in the nonrotating vierbein (2.4). On M_- , $\epsilon = 1$ (respectively -1) indicates spinors that are periodic (antiperiodic) in the vierbein (2.6). The values for a right-handed spinor are identical. Note that the stress-energy is traceless in all cases.

zero exponentially as $r \rightarrow \infty$, but numerical evidence shows that there is a range in r where this shear part is in fact the dominant part of the stress-energy.

4 Massive spinors on the \mathbb{RP}^3 geon

In this section we analyse the Hawking effect on the \mathbb{RP}^3 geon for the massive Dirac field. Subsection 4.1 recalls some properties of the geon geometry and fixes the notation, and the Boulware vacuum in the exterior region is presented in subsection 4.2. The main results are in section 4.3, where the Hartle-Hawking-like vacuum for each spin structure is constructed and analysed.

4.1 Kruskal spacetime and the \mathbb{RP}^3 geon

In the notation of [10], the Kruskal metric in the Kruskal coordinates (T, X, θ, ϕ) reads

$$ds^2 = \frac{32M^3}{r} e^{-r/(2M)} (dT^2 - dX^2) - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (4.1)$$

where $M > 0$, $T^2 - X^2 < 1$ and r is determined as a function of T and X by $T^2 - X^2 = 1 - r/(2M)$. The manifold consists of the right and left exteriors, denoted respectively by R and L , and the future and past interiors, denoted respectively by F and P , separated from each other by the bifurcate Killing horizon at $|T| = |X|$. The four regions are individually covered by the Schwarzschild coordinates (t, r, θ, ϕ) , in which the metric reads

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (4.2)$$

where $2M < r < \infty$ in the exteriors and $0 < r < 2M$ in the interiors. The coordinate transformation in R is

$$\begin{aligned} T &= \left(\frac{r}{2M} - 1 \right)^{\frac{1}{2}} e^{r/(4M)} \sinh \left(\frac{t}{4M} \right), \\ X &= \left(\frac{r}{2M} - 1 \right)^{\frac{1}{2}} e^{r/(4M)} \cosh \left(\frac{t}{4M} \right), \end{aligned} \quad (4.3)$$

and the transformations in the other regions are given in [25]. The exteriors are static, with the timelike Killing vector ∂_t .

The \mathbb{RP}^3 geon is the quotient of the Kruskal manifold under the \mathbb{Z}_2 isometry group generated by the map

$$J : (T, X, \theta, \phi) \mapsto (T, -X, \pi - \theta, \phi + \pi). \quad (4.4)$$

The construction is analogous to that of M_- from M_0 in subsection 2.1. Further discussion, including conformal diagrams, can be found in [10].

As the Kruskal manifold has spatial topology $\mathbb{R} \times S^2$, it is simply connected and has a unique spin structure. The quotient construction implies [26] that the geon has fundamental group \mathbb{Z}_2 and admits two spin structures. As in section 2, we describe these spin structures in terms of periodic and antiperiodic boundary conditions for spinors in a specified vierbein. On Kruskal, a standard reference vierbein is

$$\begin{aligned} V_0 &= \partial_T & V_1 &= \partial_X \\ V_2 &= \partial_\theta & V_3 &= \partial_\phi. \end{aligned} \quad (4.5)$$

A second useful vierbein is

$$\begin{aligned} V_0 &= \partial_T \\ V_1 &= \cos \phi \partial_X + \sin \phi \partial_\theta \\ V_2 &= -\sin \phi \partial_X + \cos \phi \partial_\theta \\ V_3 &= \partial_\phi, \end{aligned} \quad (4.6)$$

which rotates by π in the (X, θ) tangent plane as ϕ increases by π and is invariant under J . The vierbein (4.6) is well defined on the geon, and when the spinors are written in it, the two spin structures correspond to respectively periodic and antiperiodic boundary conditions as $\phi \mapsto \phi + \pi$. Both (4.5) and (4.6) are singular at $\theta = 0$ and $\theta = \pi$, but these coordinate singularities on the sphere can be handled by usual methods and will not affect our discussion.

In practice, we will work in the standard vierbein (4.5). The boundary conditions appropriate for the two geon spin structures will be found by the method-of-images technique of the Appendix of [23].

4.2 The Boulware vacuum

In this subsection we review the construction of the Boulware vacuum in one exterior region [27]. While this vacuum as such is well known, we will need to decompose the field in a novel basis in order to make contact with the geon in subsection 4.3.

We work in the Schwarzschild coordinates (4.2), with $r > 2M$, and in the adapted vierbein,

$$V_a^\mu = \text{diag} \left(\frac{1}{\left(1 - \frac{2M}{r}\right)^{\frac{1}{2}}}, \left(1 - \frac{2M}{r}\right)^{\frac{1}{2}}, \frac{1}{r}, \frac{1}{r \sin \theta} \right). \quad (4.7)$$

The Dirac equation (2.8) becomes

$$\left[m + \frac{\gamma^2}{ir \sin^{\frac{1}{2}} \theta} \partial_\theta \sin^{\frac{1}{2}} \theta + \frac{\gamma^3}{ir \sin \theta} \partial_\phi + \frac{\gamma^0}{i \left(1 - \frac{2M}{r}\right)^{\frac{1}{2}}} \partial_t + \frac{\gamma^1}{ir} \left(1 - \frac{2M}{r}\right)^{\frac{1}{4}} \partial_r \left(1 - \frac{2M}{r}\right)^{\frac{1}{4}} r \right] \psi = 0, \quad (4.8)$$

where the γ matrices are flat space γ 's. We adopt the representation

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\ \gamma^2 &= \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}, \end{aligned} \quad (4.9)$$

which has the advantage that charge conjugation takes the simple form

$$\psi^c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \psi^*, \quad (4.10)$$

where $*$ again stands for complex conjugation.

We use the separation ansatz

$$\psi_{\omega, k', m'}(t, r, \theta, \phi) = N \frac{e^{-i\omega t}}{r \left(1 - \frac{2M}{r}\right)^{\frac{1}{4}}} \begin{pmatrix} F(r) Y_{m'}^{k'}(\theta, \phi) \\ G(r) Y_{m'}^{k'}(\theta, \phi) \end{pmatrix}_{\omega k'}, \quad (4.11)$$

where $\omega > 0$ for modes that are positive frequency with respect to the Killing vector ∂_t and the spinor spherical harmonics $Y_{m'}^{k'}(\theta, \phi)$ are as constructed in [27]. The radial functions then satisfy

$$\left(1 - \frac{2M}{r}\right) \partial_r F - i\omega F = \left(1 - \frac{2M}{r}\right)^{\frac{1}{2}} \left(\frac{k'}{r} - im\right) G, \quad (4.12a)$$

$$\left(1 - \frac{2M}{r}\right) \partial_r G + i\omega G = \left(1 - \frac{2M}{r}\right)^{\frac{1}{2}} \left(\frac{k'}{r} + im\right) F. \quad (4.12b)$$

Following Chandrasekhar [28], we reduce the radial equations (4.12) to a pair of Schrödinger-like equations. Writing

$$\begin{pmatrix} F(r) \\ G(r) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(Z_+ + Z_-) e^{(-\frac{i}{2} \tan^{-1}(\frac{mr}{k'}))} \\ \frac{1}{2}(Z_+ - Z_-) e^{(\frac{i}{2} \tan^{-1}(\frac{mr}{k'}))} \end{pmatrix}, \quad (4.13)$$

we find that Z_{\pm} satisfy

$$\left(\frac{d}{d\hat{r}_*} \mp W \right) Z_{\pm} = i\omega Z_{\mp}, \quad (4.14)$$

where $\hat{r}^* = r^* + \frac{1}{2\omega} \tan^{-1}(\frac{mr}{k'})$, $r^* = r + 2M \ln(|r - 2M|/2M)$ and

$$W = \frac{(r^2 - 2Mr)^{\frac{1}{2}}(k'^2 + m^2r^2)^{\frac{3}{2}}}{r^2(k'^2 + m^2r^2) + \frac{k'm}{2\omega}(r^2 - 2Mr)}. \quad (4.15)$$

Z_{\pm} hence satisfy the one-dimensional wave equations

$$\left(\frac{d^2}{d\hat{r}_*^2} + \omega^2 \right) Z_{\pm} = V_{\pm} Z_{\pm}, \quad (4.16)$$

where

$$V_{\pm} = W^2 \pm \frac{dW}{d\hat{r}_*}. \quad (4.17)$$

Suppose first $\omega^2 > m^2$ in (4.16), in which case there are two linearly independent delta-normalisable solutions for each ω . One way to break this degeneracy would be to choose solutions that have the scattering theory asymptotic form,

$$\overleftarrow{Z}_{\pm} = \begin{cases} \overleftarrow{B}_{\pm} e^{-i\omega\hat{r}_*} & \hat{r}_* \rightarrow -\infty \\ e^{-i(p\hat{r}_* + \frac{Mm^2}{p} \ln(\frac{\hat{r}_*}{2M}))} + \overleftarrow{A}_{\pm} e^{i(p\hat{r}_* + \frac{Mm^2}{p} \ln(\frac{\hat{r}_*}{2M}))} & \hat{r}_* \rightarrow \infty \end{cases}, \quad (4.18a)$$

$$\overrightarrow{Z}_{\pm} = \begin{cases} e^{i\omega\hat{r}_*} + \overrightarrow{A}_{\pm} e^{-i\omega\hat{r}_*} & \hat{r}_* \rightarrow -\infty \\ \overrightarrow{B}_{\pm} e^{i(p\hat{r}_* + \frac{Mm^2}{p} \ln(\frac{\hat{r}_*}{2M}))} & \hat{r}_* \rightarrow \infty \end{cases}, \quad (4.18b)$$

where $p = \sqrt{\omega^2 - m^2}$. \overleftarrow{Z}_{\pm} is purely ingoing at the horizon and \overrightarrow{Z}_{\pm} is purely outgoing at infinity, and usual scattering theory Wronskians yield relations between the transmission and reflection coefficients. From (4.14) it further follows that $\overleftarrow{B}_+ = -\overleftarrow{B}_-$ and $\overrightarrow{A}_+ = -\overrightarrow{A}_-$, which will fix the form of a radial mode (4.13) that is purely ingoing at the horizon and a radial mode that is purely outgoing at infinity. However, to be able to handle the geon in subsection 4.3, we will need modes that transform simply under charge conjugation (4.10) and under J 4.4 when continued analytically into the F region. Using the Wronskian properties of the reflection and transmission coefficients in (4.18) and the properties of the spinor spherical harmonics [27], we find after considerable effort that a convenient set of positive frequency Boulware modes is $\{\Psi_{\omega, k', m'}^{\pm}\}$, given by

$$\Psi_{\omega, m', k'}^+ = \frac{e^{-\frac{i\pi}{2}(j+m'+(\frac{k'}{|k'|}-1)/2)} e^{-i\omega t}}{r(1 - \frac{2M}{r})^{\frac{1}{4}}} \begin{pmatrix} u(r) Y_{m'}^{k'}(\theta, \phi) \\ v(r) Y_{m'}^{k'}(\theta, \phi) \end{pmatrix}_{\omega k'}^+, \quad (4.19a)$$

$$\Psi_{\omega, m', k'}^- = \frac{e^{-\frac{i\pi}{2}(j-m'+(\frac{k'}{|k'|}-1)/2)} e^{-i\omega t}}{r(1 - \frac{2M}{r})^{\frac{1}{4}}} \begin{pmatrix} u(r) Y_{m'}^{k'}(\theta, \phi) \\ v(r) Y_{m'}^{k'}(\theta, \phi) \end{pmatrix}_{\omega k'}^-, \quad (4.19b)$$

where $\omega > m$, the radial functions with superscript $+$ are specified by the horizon asymptotic behaviour

$$\begin{pmatrix} u(r) \\ v(r) \end{pmatrix}_{\omega k'}^+ = \frac{1}{\sqrt{4\pi}} \left\{ \sqrt{1 + \sqrt{1 - |\vec{A}|^2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\omega r_*} + \frac{\vec{A}_+}{\sqrt{1 + \sqrt{1 - |\vec{A}|^2}}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i\omega r_*} \right\}, \quad \hat{r}^* \rightarrow -\infty, \quad (4.20a)$$

and

$$\begin{pmatrix} u(r) \\ v(r) \end{pmatrix}_{\omega k'}^- = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u(r) \\ v(r) \end{pmatrix}_{\omega k'}^{+*}. \quad (4.20b)$$

The key property for charge conjugation is (4.20b). The modes are complete for $\omega^2 > m^2$ and delta-orthonormal in the Dirac inner product

$$\langle \psi_1, \psi_2 \rangle = \int_{\text{angles}} \sin \theta d\theta d\phi \int_{2M}^{\infty} \frac{r^2}{(1 - \frac{2M}{r})^{\frac{1}{2}}} \psi_1^\dagger \psi_2 dr, \quad (4.21)$$

taken on a constant t hypersurface.

Suppose then $0 < \omega^2 < m^2$ in (4.16). There is now only one linearly independent delta-normalisable solution for each ω . This solution vanishes at infinity and has at the horizon the behaviour

$$Z_\pm = a_\pm \cos(\omega \hat{r}^* + \delta_\pm), \quad \hat{r}^* \rightarrow -\infty, \quad (4.22)$$

where a_\pm , and δ_\pm are real constants. Physically these solutions correspond to particles that do not reach infinity. Proceeding as above, we find that a convenient set of positive frequency Boulware modes, complete for $0 < \omega^2 < m^2$ and delta-orthonormal in the Dirac inner product (4.21), is

$$\psi_{\omega, k', m'}(t, r, \theta, \phi) = e^{-\frac{i\pi}{2}(j+|m'|+(1-\frac{k'}{|k'|})/2)} \frac{e^{-i\omega t}}{r(1 - \frac{2M}{r})^{\frac{1}{4}}} \begin{pmatrix} F(r) Y_{m'}^{k'}(\theta, \phi) \\ G(r) Y_{m'}^{k'}(\theta, \phi) \end{pmatrix}_{\omega k'}, \quad (4.23)$$

where $0 < \omega < m$ and the radial functions are specified by the horizon asymptotic behaviour

$$\begin{pmatrix} F(r) \\ G(r) \end{pmatrix}_{\omega k'} = \frac{1}{\sqrt{2\pi}} \left\{ \begin{pmatrix} e^{i\delta_+} \\ 0 \end{pmatrix} e^{i\omega r_*} + \begin{pmatrix} 0 \\ e^{-i\delta_+} \end{pmatrix} e^{-i\omega r_*} \right\}, \quad \hat{r}^* \mapsto -\infty. \quad (4.24)$$

Up to this point we have used the Schwarzschild vierbein (4.7). To make contact with the geon in subsection 4.3, we need to express the modes in a vierbein that is regular at the horizons. We therefore now transform our modes to the Kruskal vierbein (4.5) by the spinor transformation $\psi \mapsto e^{\frac{k}{8M}\gamma^0\gamma^1}\psi$. We suppress the explicit transformed expressions and continue to use the same symbols for the mode functions.

We are now ready to quantise. The field is expanded in our orthonormal modes and their charge conjugates as

$$\begin{aligned}\Psi = & \sum_{m',k'} \int_0^m d\omega \left(a_{\omega,m',k'} \psi_{\omega,m',k'} + b_{\omega,m',k'}^\dagger \psi_{\omega,m',k'}^c \right) \\ & + \sum_{m',k'} \int_m^\infty d\omega \left(a_{+,\omega,m',k'} \Psi_{\omega,m',k'}^+ + a_{-,\omega,m',k'} \Psi_{\omega,m',k'}^- \right. \\ & \left. + b_{+,\omega,m',k'}^\dagger \Psi_{\omega,m',k'}^{+,c} + b_{-,\omega,m',k'}^\dagger \Psi_{\omega,m',k'}^{-,c} \right) , \quad (4.25)\end{aligned}$$

where the annihilation and creation operators satisfy the usual anticommutation relations. The vacuum annihilated by the annihilation operators is the Boulware vacuum $|0_B\rangle$. $|0_B\rangle$ is by construction the state void of particles with respect to the Schwarzschild Killing time.

4.3 The Hartle-Hawking like vacuum and Bogolubov transformation on the geon

In this subsection we decompose the geon Hartle-Hawking-like vacuum into Boulware excitations. We use the analytic continuation method, following closely subsection 2.4.

The Hartle-Hawking vacuum on Kruskal is defined by mode functions that are purely positive frequency with respect to the horizon generators and hence analytic in the lower half-plane in the complexified Kruskal time T . It follows that on Kruskal we can construct W -modes whose vacuum is the Hartle-Hawking vacuum by analytically continuing the Boulware-modes across the horizons in the lower half-plane in T . The quotient from Kruskal to the geon defines in each spin structure on the geon the Hartle-Hawking-like vacuum $|0_G\rangle$, by restriction to the Kruskal W -modes that are invariant under the map J (4.4). Our task is to find these modes.

Let us denote by W^F the restriction of the sought-for W -modes to the region F and by ψ^F the analytic continuation of Boulware modes from R to F , expressed in the rotating Kruskal vierbein (4.6). As this vierbein is invariant under J , we can set

$$W^F(T, X, \theta, \phi) = \psi^F(T, X, \theta, \phi) + \epsilon \psi^F(T, -X, \pi - \theta, \phi + \pi) , \quad (4.26)$$

where $\epsilon = 1$ (respectively -1) for spinors that are periodic (antiperiodic) in this vierbein. As the transformation from this vierbein to the standard Kruskal vierbein (4.5) is a rotation by $-\pi$ in the (X, θ) tangent plane as ϕ increases by π , the corresponding spinor transformation is

$$W^F(T, X, \theta, \phi) \mapsto W_s^F(T, X, \theta, \phi) := e^{\frac{\phi \gamma^1 \gamma^2}{2}} W^F(T, X, \theta, \phi) , \quad (4.27)$$

where the subscript s refers to the standard vierbein (4.5). In the standard vierbein, (4.26) hence becomes

$$W_s^F(T, X, \theta, \phi) = \psi_s^F(T, X, \theta, \phi) + \epsilon e^{-\frac{\pi \gamma^1 \gamma^2}{2}} \psi_s^F(T, -X, \pi - \theta, \phi + \pi) . \quad (4.28)$$

We can therefore employ the condition (4.28). It can be verified that the functions have the correct transformation properties also in the P region of Kruskal.

The computations are lengthy but straightforward, using the explicit coordinate transformation (4.3) and the near-horizon behaviour in (4.20) and (4.24). Continued back to R , we find that the W -modes with $\omega > m$ are

$$\begin{aligned} W_{\omega,k',m'}^+(t, r, \theta, \phi) &= \frac{1}{\sqrt{2 \cosh(4\pi M\omega)}} \left(e^{2\pi M\omega} \Psi_{\omega,k',m'}^+ + \epsilon e^{-2\pi M\omega} \Psi_{\omega,k',-m'}^{-,c} \right) , \\ W_{\omega,k',m'}^-(t, r, \theta, \phi) &= \frac{1}{\sqrt{2 \cosh(4\pi M\omega)}} \left(e^{2\pi M\omega} \Psi_{\omega,k',m'}^- - \epsilon e^{-2\pi M\omega} \Psi_{\omega,k',-m'}^{+,c} \right) , \end{aligned} \quad (4.29)$$

and those with $0 < \omega < m$ are

$$W_{\omega,k',m'}(t, r, \theta, \phi) = \frac{1}{\sqrt{2 \cosh(4\pi M\omega)}} \left(e^{2\pi M\omega} \psi_{\omega,k',m'} + \epsilon \frac{m'}{|m'|} e^{-2\pi M\omega} \psi_{\omega,k',-m'}^c \right) . \quad (4.30)$$

These modes are appropriately delta-orthonormal in the Dirac inner product (4.21) in R and hence also orthonormal in the Dirac inner product on the geon. It can be verified that the factors $m'/|m'|$ appearing in (4.30) cannot be absorbed into the phase factors of the Boulware modes.

On the geon, the expansion of the field in the W -modes reads

$$\begin{aligned} \Psi &= \sum_{m',k'} \int_0^m d\omega \left(c_{\omega,m',k'} W_{\omega,m',k'} + d_{\omega,m',k'}^\dagger W_{\omega,m',k'}^c \right) \\ &+ \sum_{m',k'} \int_m^\infty d\omega \left(c_{+,\omega,m',k'} W_{\omega,m',k'}^+ + c_{-,\omega,m',k'} W_{\omega,m',k'}^- \right. \\ &\quad \left. + d_{+,\omega,m',k'}^\dagger W_{\omega,m',k'}^{+,c} + d_{-,\omega,m',k'}^\dagger W_{\omega,m',k'}^{-,c} \right) , \end{aligned} \quad (4.31)$$

with the usual anticommutation relations, and $|0_G\rangle$ is the state annihilated by all the annihilator operators. Comparison of (4.25) and (4.31) gives the Bogolubov transformation

$$\begin{aligned} a_{\omega,k',m'} &= \frac{1}{\sqrt{2 \cosh(4\pi M\omega)}} \left(e^{2\pi M\omega} c_{\omega,k',m'} + \epsilon \frac{m'}{|m'|} e^{-2\pi M\omega} d_{\omega,k',-m'}^\dagger \right) , \\ b_{\omega,k',m'}^\dagger &= \frac{1}{\sqrt{2 \cosh(4\pi M\omega)}} \left(e^{2\pi M\omega} d_{\omega,k',m'}^\dagger + \epsilon \frac{m'}{|m'|} e^{-2\pi M\omega} c_{\omega,k',-m'} \right) , \\ a_{+,\omega,k',m'} &= \frac{1}{\sqrt{2 \cosh(4\pi M\omega)}} \left(e^{2\pi M\omega} c_{+,\omega,k',m'} - \epsilon e^{-2\pi M\omega} d_{-,\omega,k',-m'}^\dagger \right) , \\ a_{-,\omega,k',m'} &= \frac{1}{\sqrt{2 \cosh(4\pi M\omega)}} \left(e^{2\pi M\omega} c_{-,\omega,k',m'} + \epsilon e^{-2\pi M\omega} d_{+,\omega,k',-m'}^\dagger \right) , \\ b_{+,\omega,k',m'}^\dagger &= \frac{1}{\sqrt{2 \cosh(4\pi M\omega)}} \left(e^{2\pi M\omega} d_{+,\omega,k',m'}^\dagger - \epsilon e^{-2\pi M\omega} c_{-,\omega,k',-m'} \right) , \\ b_{-,\omega,k',m'}^\dagger &= \frac{1}{\sqrt{2 \cosh(4\pi M\omega)}} \left(e^{2\pi M\omega} d_{-,\omega,k',m'}^\dagger + \epsilon e^{-2\pi M\omega} c_{+,\omega,k',-m'} \right) . \end{aligned} \quad (4.32)$$

It follows that the expansion of $|0_G\rangle$ in the Boulware-excitations is

$$\begin{aligned}
|0_G\rangle = & \prod_{\substack{0 < \omega < m \\ m', k'}} \frac{1}{(e^{-8\pi M\omega} + 1)^{\frac{1}{2}}} \sum_{q=0,1} \left(\frac{\epsilon m'}{|m'|} \right)^q e^{-4\pi M\omega q} (a_{\omega, k', m'}^\dagger b_{\omega, k', -m'}^\dagger)^q |0_B\rangle \\
& \times \prod_{\substack{\omega > m \\ m', k'}} \frac{1}{(e^{-8\pi M\omega} + 1)^{\frac{1}{2}}} \sum_{q=0,1} (-\epsilon)^q e^{-4\pi M\omega q} |q\rangle_{+, \omega, m', k'} |q\rangle_{-, \omega, -m', k'} , \quad (4.33)
\end{aligned}$$

where

$$\begin{aligned}
|q\rangle_{+, \omega, m', k'} &= a_{+, \omega, m', k'}^\dagger |0_B\rangle , \\
|q\rangle_{-, \omega, m', k'} &= b_{-, \omega, m', k'}^\dagger |0_B\rangle . \quad (4.34)
\end{aligned}$$

A comparison of (4.33) and (2.48) shows that $|0_G\rangle$ is closely similar to the state $|0_-\rangle$ on M_- , and the discussion at the end of subsection 2.4 adapts directly here. $|0_G\rangle$ does not appear thermal to generic static observers in R , but it appears thermal in the standard Hawking temperature $(8\pi M)^{-1}$ near the infinity when probed by operators that only couple to one member of each correlated pair in (4.33), such as operators that only couple to a definite sign of the angular momentum quantum number m' . In particular, number operator expectation values are thermal, and the isometry arguments of [10] show that the experiences of any static observer become asymptotically thermal in the large $|t|$ limit.

The explicit appearance of ϵ in (4.33) shows that the nonthermal correlations in $|0_G\rangle$ reveal the geon spin structure to an observer in R . This is a phenomenon that could not have been anticipated just from the geometry of R , which in its own right has only one spin structure.

5 Discussion

This paper has discussed thermal effects for the free Dirac field on the \mathbb{RP}^3 geon and on a topologically analogous flat spacetime M_- via a Bogolubov transformation analysis. Compared with the scalar field [10], the main new issue with fermions is that the spacetimes admit two inequivalent spin structures, and there are hence two inequivalent Hartle-Hawking like vacua on the geon and two inequivalent Minkowski-like vacua on M_- . We showed that an observer in the exterior region of the geon can detect both the nonthermality of the Hartle-Hawking like state and the spin structure of this state by suitable interference measurements, and similar results hold for a Rindler observer on M_- . When probed with suitably restricted operators, such as operators at asymptotically late Schwarzschild (respectively Rindler) times, these states nevertheless appear thermal in the usual Hawking (Unruh) temperature, for the same geometric reasons as in the scalar case [10]. We further computed the stress-energy expectation value on M_- in the massless limit, showing that the two spin structures are distinguished by the sign of a nonvanishing shear component. As a by-product of the analysis, we presented the Bogolubov transformation for the Unruh effect for the massive Dirac field in $(3+1)$ -dimensional Minkowski space, complementing and correcting the previous literature.

It would be interesting to explore how to observe the nonthermal correlations in the Hartle-Hawking like and Minkowski-like states by particle detectors that couple to the field in some local fashion. As the states in question are not invariant under the locally-defined Killing vectors with respect to which the thermal properties arise, the deviations from thermality would need to be analysed in a setting that can handle time-dependent detector response [22].

As a late time observer in the geon exterior sees a thermal state in the usual Hawking temperature, the classical laws of black hole mechanics lead the observer to assign to the geon the same entropy as to a conventional Schwarzschild hole with the same mass. It was found in [10] that an attempt to evaluate the geon entropy by path-integral methods leads to half of the Bekenstein-Hawking entropy of a Schwarzschild hole of the same mass, and it was suggested that state-counting computations of the geon entropy could shed light on this discrepancy. Our work says little of what the full framework of such a computation would be, but our work would presumably provide part of the fermionic machinery in the computation. In particular, the issue of the spin structure would need to be faced seriously: Does an entropy computation by state-counting need to count the two spin structures as independent degrees of freedom?

Acknowledgements

I would like to thank Jorma Louko for many useful discussions and for reading the manuscript. Thanks also to Edward Armour and John Barrett for useful comments and to Daniele Oriti for correspondence on reference [17]. Comments from an anonymous referee helped improve the presentation. This work was supported by the University of Nottingham.

References

- [1] S. W. Hawking, *Commun. Math. Phys.* **43**, 199 (1975)
- [2] J. D. Bekenstein, *Phys. Rev. D* **7**, 2333 (1973)
- [3] J. B. Hartle and S. W. Hawking, *Phys. Rev. D* **13**, 2188 (1976)
- [4] W. Israel, *Phys. Lett.* **57A**, 107 (1976)
- [5] W. G. Unruh, *Phys. Rev. D* **14**, 870 (1976)
- [6] G. W. Gibbons and S. W. Hawking, *Phys. Rev. D* **15**, 2752 (1977)
- [7] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982)
- [8] B. S. Kay and R. M. Wald, *Phys. Rep.* **207**, 49 (1991)
- [9] R. M. Wald, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics* (The University of Chicago Press, Chicago, 1994).
- [10] J. Louko and D. Marolf, *Phys. Rev. D* **58**, 024007 (1998)

- [11] J. L. Friedman, K. Schleich and D. M. Witt, Phys. Rev. Lett. **71**, 1486 (1993):
Erratum, Phys. Rev. Lett. **75**, 1872 (1995)
- [12] C. W. Misner and J. A. Wheeler, Ann. Phys. **2**, 525 (1957)
- [13] D. Giulini, Ph.D. thesis (University of Cambridge, 1989)
- [14] S. Takagi, Prog. Theor. Phys. Supp. **88**, 1 (1986)
- [15] P. Candelas and D. Deutsch, Proc. R. Soc. Lond. A **362**, 251 (1978)
- [16] M. Soffel, B. Muller and W. Greiner, Phys. Rev. D **22**, 1935 (1979)
- [17] D. Oriti, Nuovo Cim. **B115**, 1005 (2000)
- [18] S. J. Avis and C. J. Isham, in *Recent Developments in Gravitation-Cargese 1978* ed M. Levy and S. Deser (Plenum Press, New York, 1979)
- [19] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill Book Company, 1964)
- [20] C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1975)
- [21] D. Zwillinger, *CRC Standard Mathematical Tables and Formulae* (Chapman and Hall/CRC Press LLC, 2003)
- [22] S. Schlicht, E-print gr-qc/0306022
- [23] R. Banach and J. S. Dowker, J. Phys. A: Math. Gen. **12**, 2545 (1979)
- [24] M. R. Spiegel, *Complex Variables* (McGraw-Hill, New York, 1972)
- [25] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973)
- [26] A. Hatcher, *Algebraic Topology* (Cambridge University Press, Cambridge, 2002), Proposition 1.40 and Exercise 1.3.24.
- [27] D. G. Boulware, Phys. Rev. D **12**, 350 (1975)
- [28] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, New York, 1985)